A COMPLEX FROBENIUS THEOREM, MULTIPLIER IDEAL SHEAVES AND HERMITIAN-EINSTEIN METRICS ON STABLE BUNDLES

Ben Weinkove
Department of Mathematics, Columbia University
New York, NY 10027
Email: *weinkove@math.columbia.edu*

1. Introduction

In [Ko], J.J. Kohn introduced the notion of 'subelliptic multipliers' in his work on the $\overline{\partial}$-Neumann problem on pseudo-convex domains in order to prove subelliptic estimates on the boundary under certain conditions. He believed that his techniques would have wide applications in deriving estimates by algebraic methods. Following this line of approach, Nadel [Na] defined the *multiplier ideal sheaf*, and used it to prove the existence of Kähler-Einstein metrics on certain Fano manifolds with symmetries (his proofs were later simplified in [DeKo].) Given a plurisubharmonic function ϕ on the manifold, the multiplier ideal sheaf of ϕ is the sheaf of germs of holomorphic functions f satisfying

$$\int |f|^2 e^{-\phi} < \infty.$$

The sheaf measures the singularities of ϕ. The concept of the multiplier ideal sheaf, which had already been considered implicitly in the work of Bomberi [Bo], Skoda [Sk] and Siu [S1], is an important one in algebraic geometry, and has led to some important breakthroughs, such as the proof by Siu [S3] of the big Matsusaka theorem and effective results on global generation and very ampleness (e.g. [S4], [De].)

In a number of lectures and seminars ([S5] for example), Y-T. Siu has advocated that generalized multiplier ideal sheaves have many applications to problems in partial differential equations and geometry. A multiplier ideal sheaf should measure the position and extent of the failure of an *a priori* estimate. A key concept is that the sheaf $\mathcal{I}$ should satisfy a differential inclusion relation which arises from known estimates. In Kohn's case ([Ko], Proposition 4.7.(G)), the differential relation takes the form

$$\partial \mathcal{I} \subset \mathcal{M},$$

where $\mathcal{M}$ is Kohn's 'multiplier module'. In this paper, we look at the notion of multiplier ideal sheaves in the context of the problem of the existence of Hermitian-Einstein metrics on stable bundles over compact Kähler manifolds. It is our hope that this may provide some clues as to how to proceed in other situations. The problem of existence of Hermitian-Einstein metrics has been solved over curves by Narasimhan and Seshadri [NS] (see also [D1] for an alternative proof), over algebraic surfaces by Donaldson [D2] and over general compact Kähler manifolds by Uhlenbeck and Yau [UY]. In [UY], using a continuity method, it is shown that proving the existence of an Hermitian-Einstein metric can be reduced to proving a certain *a priori* estimate. If this estimate fails to hold then they show that there exists a *destabilizing subsheaf* which contradicts the assumption that the original bundle was stable.

We give an alternative construction of the destabilizing subsheaf in the case of algebraic surfaces, following the ideas of Siu. Instead of the continuity method, we use the Donaldson heat flow (see section 3) which is roughly equivalent. The crucial estimate is the C^0 estimate for this flow of metrics. Abusing terminology, we define a 'multiplier ideal sheaf' picking out those directions in which the metric does not blow up. We show that, using a bound on the contracted curvature ΛF along the flow, the multiplier ideal sheaf $\mathcal{F}$ satisfies the differential inclusion relation

$$\overline{\partial}\mathcal{F} \subset \mathcal{F} \otimes T^{0,1}.$$

The key property that the sheaf must satisfy is coherence. We show this using the differential inclusion relation and a complex Frobenius theorem which is given as follows.

Let E be a holomorphic vector bundle of rank r over a complex manifold X of complex dimension n. Denote the 'partial connection' associated to the holomorphic structure on E by $\overline{\partial}$. Denote by $\mathcal{O}$, $\mathcal{O}_R$ and $\mathcal{C}^\infty$ the sheaves of holomorphic, complex-valued real analytic and smooth functions on X respectively. Let $\mathcal{E}$, $\mathcal{E}_R$ and $\mathcal{E}^\infty$ denote the sheaves of holomorphic, real analytic and C^∞ sections of E respectively. $T^{0,1}$ will denote the corresponding sheaf of (0,1) forms on X.

Theorem 1 *Let $\mathcal{F}_R \subseteq \mathcal{E}_R$ be a subsheaf satisfying*

(i) $\mathcal{F}_R$ *is locally finitely generated;*

(ii) $\overline{\partial}\mathcal{F}_R \subseteq \mathcal{F}_R \otimes T^{0,1}$.

Then $\mathcal{F}_R$ is locally finitely generated by holomorphic sections of E. Denote by $\mathcal{F}$ the sheaf of $\mathcal{O}$-modules $\ker \overline{\partial}|_{\mathcal{F}_R}$. *It follows that $\mathcal{F} \subset \mathcal{E}$ is coherent and*

$$\mathcal{F} \otimes \mathcal{O}_R = \mathcal{F}_R. \tag{1.1}$$

The corresponding result holds for a subsheaf $\mathcal{F}^\infty \subseteq \mathcal{E}^\infty$ if condition (i) is replaced by

(i)' For each x in X, there exists an open set U containing x and a free resolution

$$\cdots \longrightarrow (\mathcal{C}_U^\infty)^{d_2} \longrightarrow (\mathcal{C}_U^\infty)^{d_1} \longrightarrow (\mathcal{C}_U^\infty)^{d_0} \longrightarrow \mathcal{F}_U^\infty \longrightarrow 0,$$

for $d_i \geq 0$.

If the subsheaf is locally free, this follows from the Koszul-Malgrange integrability theorem for vector bundles [KM], which is a linear version of the Newlander-Nirenberg theorem for integrable almost complex structures [NN].

A result similar to the C^∞ case of Theorem 1 has been proved by very different methods in a recent preprint of Pali [Pa]. His theorem applies to sheaves of smooth functions with a $\overline{\partial}$ operator satisfying $\overline{\partial}^2 = 0$, and which admit local free resolutions of finite length. Pali does not assume that the sheaf is a subsheaf of a holomorphic vector bundle. In the case that it is, condition *(i)'* of Theorem 1 is slightly weaker than his condition, since we do not assume that the resolution is of finite length.

Note that the differential inclusion relation *(ii)* is not enough by itself to prove coherence. This can be seen by the following counterexample. Fix an open set U in X and consider the subsheaf with stalk $\mathcal{E}_{R,x}$ ($\mathcal{E}_x^\infty$) for x in U and the zero stalk otherwise. This satisfies condition *(ii)* but is clearly not coherent.

In our application of Theorem 1, we will only use the real analytic case. We define our multiplier ideal sheaf as follows. Suppose that X is now a compact Kähler manifold, and let $\tilde{X}$ be an open subset of X. Let $\{H_k\}_{k=0}^\infty$ be a sequence of metrics on E over $\tilde{X}$, and define endomorphisms $h_k = H_0^{-1} H_k$. Define a presheaf $\mathcal{P}_\mathcal{R} \subseteq \mathcal{E}_R$ over $\tilde{X}$ by setting the sections over an open set $U \subset \tilde{X}$ to be

$$\mathcal{P}_\mathcal{R}(U) = \{s \in \mathcal{E}_R(U) \mid \int_U |s|^2_{H_k} \to 0 \text{ as } k \to \infty\},$$

where integration is performed with respect to the volume form induced from the Kähler metric on X. Then let $\mathcal{F}_R$ be the sheaf associated to the presheaf $\mathcal{P}_{\mathcal{R}}$. We will call $\mathcal{F}_R$ the *multiplier ideal sheaf* associated to the sequence of metrics $\{H_k\}$. (For a different definition of a multiplier ideal sheaf for bundles, see [DC].) If the sequence of metrics satisfies certain conditions, then we can prove the differential inclusion relation and apply Theorem 1 to obtain a coherent subsheaf of E.

Theorem 2 *Suppose that there exists a positive constant C and a real analytic endomorphism h_∞ of E such that the sequence of metrics $\{H_k\}_{k=0}^{\infty}$ satisfies*

(i) $H_k \leq H_0$ *for all* k*;*

(ii) $i\hat{F}_{H_k} H_0 \leq C H_0$ *for all* k*;*

(iii) $h_k \to h_\infty$ *uniformly on compact subsets of* $\tilde{X}$*, for* $h_k = H_0^{-1} H_k$*.*

Then the multiplier ideal sheaf $\mathcal{F}_R$ associated to $\{H_k\}$ defines a holomorphic coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ over $\tilde{X}$.

We will see how to use Theorem 2 to construct the destabilizing subsheaf in a proof of the theorem of Donaldson [D2]:

Theorem 3 *Let $X \subset \mathbf{CP}^N$ be a projective algebraic surface and let ω be the Kähler metric on X induced from the Fubini-Study metric on $\mathbf{CP}^n$. Let E be a holomorphic vector bundle of rank r over X which is stable with respect to the metric ω. Then E admits a Hermitian-Einstein metric.*

It should be emphasized that the proof of Theorem 3 given here differs from existing proofs only in the construction of the destabilizing subsheaf and does not provide a simplification (in fact the given proof may be easily shortened.) However, the method should help elucidate a certain point of view and provide a useful model for the further study of multiplier ideal sheaves. We use the Yang-Mills heat flow and the associated flow of metrics, taking estimates from [D2], [DoKr] and [UY]. The heat flow method has also been used by Simpson [Si] and de Bartolomeis and Tian [DT] to prove generalizations of the Theorem of Donaldson-Uhlenbeck-Yau.

The paper is arranged as follows. In section 2 we prove the complex Frobenius theorem. In section 3 we prove Theorem 2 by making use of the

differential inclusion relation. In section 4 we give a proof of Theorem 3, using the Yang-Mills flow and Theorem 2.

Notation

We will use the notation from [D2]. In particular, the Kähler form ω on X can be written in normal coordinates at a point as

$$\omega = \frac{1}{2} i \sum_{j=1}^{2} dz^j \wedge d\overline{z}^j,$$

and the operator Λ acts on $(1,1)$ forms at this point by

$$\Lambda\left(\sum_{j,k=1}^{2} a_{j\overline{k}} dz^j \wedge d\overline{z}^k\right) = -2i \sum_{j=1}^{2} a_{j\overline{j}}.$$

If ϕ and ψ are form-valued sections of E, define

$$\langle \phi, \psi \rangle_H = \sum_{\alpha,\beta=1}^{r} H_{\alpha\overline{\beta}} \phi^\alpha \wedge \overline{\psi^\beta}.$$

Note that if H_0 and H are metrics on E, and $h = H_0^{-1} H$, then the associated curvatures F_{H_0} and F_H differ by

$$F_H = F_{H_0} + \overline{\partial}(h^{-1} \partial_0 h).$$

For convenience, write $\hat{F}$ for ΛF.

2. Complex Frobenius theorem

In this section we give a proof of Theorem 1. We begin with the real analytic case. The statement is local so it will suffice to prove the following. Let $\mathcal{F}_R$ be a subsheaf of $\mathcal{O}_R^r$ over the polydisc $U_\eta = \{z \in \mathbf{C}^n \mid |z_j| < \eta, j = 1, \ldots, n\}$ for some $\eta > 0$ satisfying the hypotheses (i) and (ii) of Theorem 1. Let $f_1, \ldots, f_q$ be sections of $\mathcal{F}_R$ over U_η generating the stalks $\mathcal{F}_{R,x}$ for all x in U_η. We will show that there exists some η' with $0 < \eta' < \eta$ and sections $g_1, \ldots, g_q$ of $\mathcal{F}_R$ over $U_{\eta'}$ such that $\overline{\partial} g_i = 0$ for $i = 1, \ldots, q$, with the g_i generating $\mathcal{F}_{R,x}$ for all x in $U_{\eta'}$.

We will give a proof by induction on the number of variables $z_1, \ldots, z_p$ in which the generators are holomorphic.

Step 1. We begin by first showing that for small enough η' there exist sections $g_1, \ldots, g_q$ over $U_{\eta'}$ satisfying $\partial_{\overline{1}} g_i = 0$, and such that $g_1, \ldots, g_q$ generate $\mathcal{F}_{R,x}$ for all x in $U_{\eta'}$. By hypothesis, there exists a $q \times q$ matrix of real analytic functions A_1 on U_η satisfying

$$\begin{pmatrix} \partial_{\overline{1}} f_1 \\ \vdots \\ \partial_{\overline{1}} f_q \end{pmatrix} = A_1 \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix}.$$

We will show that for small enough η' there exists a real analytic map $B : U_{\eta'} \to GL(q, \mathbf{C})$ satisfying

$$\partial_{\overline{1}} B + BA = 0. \tag{2.1}$$

To see that this will give the desired conclusion, set

$$\begin{pmatrix} g_1 \\ \vdots \\ g_q \end{pmatrix} = B \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix}.$$

Since B is invertible, the $g_1, \ldots, g_q$ generate $\mathcal{F}_{R,x}$ for x in $U_{\eta'}$ and differentiating the above gives

$$\begin{pmatrix} \partial_{\overline{1}} g_1 \\ \vdots \\ \partial_{\overline{1}} g_q \end{pmatrix} = (\partial_{\overline{1}} B) \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix} + BA_1 \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix} = 0.$$

We will pull back the f_i via the dilation map $\delta_r : \mathbf{C}^n \to \mathbf{C}^n$ given by $(z_1, z_2, \ldots, z_n) \mapsto (rz_1, z_2, \ldots, z_n)$ for some $r < 1$. Notice that A_1 scales by the factor r. We will solve the problem for these new f_i and then transform back.

Denote by D_η the open disc of radius η in $\mathbf{C}$. Multiply A_1 by a smooth cut-off function $\psi = \psi(|z_1|)$ with compact support in $D_{\eta/r}$ and equal to 1 on $D_{\eta/2r}$. We can then regard A_1 as a function on $\mathbf{C}$ with parameters $z_2, \overline{z}_2, \ldots, z_n, \overline{z}_n$ varying analytically in a small polycylinder which may be shrunk if necessary. Replacing these variables by $\zeta_1, \ldots, \zeta_{2n-2}$, we can view A_1 as having holomorphic parameters ζ_i. A_1 is smooth on $\mathbf{C}$, has compact support in $D_{\eta/r}$ and is real analytic in $D_{\eta/2r}$. By choosing r small enough, we can suppose that, for a fixed $0 < \epsilon < 1$, the norm $\|A_1\|_{C^\epsilon}$ is as small as we like.

Now set $B = I + F$, so that equation (2.1) becomes

$$\partial_{\overline{1}} F + (I + F)A_1 = 0. \tag{2.2}$$

We will use the following implicit function theorem to solve (2.2) for small real analytic F.

Theorem 2.1 *Let Y_1, Y_2 and Z be Banach spaces, and set $Y = Y_1 \times Y_2$. Let $\Phi : Y \to Z$ be a smooth (holomorphic) map. Fix a point (y_1, y_2) in Y. If the partial derivative $(D_2\Phi)(y_1, y_2) : Y_2 \to Z$ is surjective and admits a bounded right inverse $P : Z \to Y_2$, then there is a smooth (holomorphic) map f from a neighbourhood of y_1 in Y_1 to a neighbourhood of y_2 in Y_2 such that*

$$\Phi(\eta, f(\eta)) = \Phi(y_1, y_2).$$

It follows that if η_z is a family of elements of Y_1 in the above neighbourhood of y_1, varying smoothly (holomorphically) in a parameter z, then $f(\eta_z)$ will also vary smoothly (holomorphically) in z.

Apply the theorem with Y_1 and Z the Banach spaces of $q \times q$ matrix-valued functions on $\mathbf{C}$ with the norm $\| \ \|_{C^\epsilon}$ bounded, and Y_2 the same space with norm $\| \ \|_{C^{1+\epsilon}}$. Take $(y_1, y_2) = (0, 0)$ and

$$\Phi(A_1, F) = \partial_{\overline{1}} F + (I + F)A_1.$$

Then $(D_2\Phi)(y_1, y_2) : Y_2 \to Z$ is given by

$$(D_2\Phi)(y_1, y_2)(\alpha) = \partial_{\overline{1}}\alpha,$$

which has bounded right inverse $P : Z \to Y_2$ given by

$$P(\beta)(z) = \frac{1}{2\pi i}\int_{\mathbf{C}} \frac{\beta(w)}{w - z} dw \wedge d\overline{w}.$$

Hence, if $\|A_1\|_{C^\epsilon}$ is small enough then there exists $F = F(A_1)$ with $\|F\|_{C^{1+\epsilon}}$ small satisfying equation (2.2). Now since A_1 depends holomorphically on the parameters ζ_i, and since Φ is holomorphic, the solution F will also be holomorphic in these parameters. By making the norm of A_1 smaller if necessary, the solution $B = I + F$ will be invertible. Moreover, since F is a solution of the elliptic equation

$$(\partial_{\overline{z}_1} + \partial_{\overline{\zeta}_1} + \ldots + \partial_{\overline{\zeta}_{2n-2}})F + (I + F)A_1 = 0,$$

it follows that F must also be analytic in the variables z_1 and $\overline{z}_1$ by the theorem that sufficiently smooth solutions to elliptic equations with real analytic coefficients are themselves real analytic. Replacing the $\zeta_1, \ldots, \zeta_{2n-2}$ with $z_2, \ldots, \overline{z}_n$, we see that F is analytic in $z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n$. We have now completed the first step in the induction.

Step 2. Fix p with $1 \leq p \leq n-1$. Our inductive hypothesis is that for η' small enough, there exist sections $f_1, \ldots, f_q$ over $U_{\eta'}$ satisfying $\partial_{\overline{j}} f_i = 0$ for $j = 1, \ldots, p$ and generating $\mathcal{F}_{R,x}$ for x in $U_{\eta'}$. We will show that, for some $\eta'' > 0$, there exist sections $g_1, \ldots, g_q$ over $U_{\eta''}$ satisfying $\partial_{\overline{j}} g_i = 0$ for $j = 1, \ldots, p+1$ and such that $g_1, \ldots, g_q$ generate $\mathcal{F}_{R,x}$ for x in $U_{\eta''}$.

We know that there exist real analytic functions of $q \times q$ matrices A_k on $U_{\eta'}$ for $k = 1, \ldots, n$ satisfying

$$\begin{pmatrix} \partial_{\overline{k}} f_1 \\ \vdots \\ \partial_{\overline{k}} f_q \end{pmatrix} = A_k \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix}. \tag{2.3}$$

The inductive hypothesis implies that $A_k = 0$ for $k = 1, \ldots, p$. We will show that, for η'' small enough, there exists a real analytic function $B : U_{\eta''} \to GL(q, \mathbf{C})$, holomorphic in the variables $z_1, \ldots, z_p$, satisfying

$$\partial_{\overline{p+1}} B + B A_{p+1} = 0.$$

Then, as before, the sections g_i given by

$$\begin{pmatrix} g_1 \\ \vdots \\ g_q \end{pmatrix} = B \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix}$$

will satisfy the required properties.

Before using the implicit function theorem, we must ensure that A_{p+1} is holomorphic in $z_1, \ldots, z_p$. Notice that in equation (2.3) for $k = p+1$, A_{p+1} is an analytic function of all of the variables $z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n$ whereas the $\partial_{\overline{p+1}} f_i$ and the f_i do not depend on $\overline{z}_1, \ldots, \overline{z}_p$. Setting $\overline{z}_1 = \ldots = \overline{z}_p = 0$ we get an A_{p+1} which satisfies (2.3) and is holomorphic in $z_1, \ldots, z_p$.

We can now finish the proof. Let $B = I + F$. We use the same method as before to solve

$$\partial_{\overline{p+1}} F + (I + F) A_{p+1} = 0$$

for F small, real analytic, and holomorphic in $z_1, \ldots, z_p$, given that A_{p+1} is holomorphic in $z_1, \ldots, z_p$. After dilating the z_{p+1} coordinate and multiplying by a cut-off function $\psi = \psi(|z_{p+1}|)$, we can take A_{p+1} to be a smooth function on $\mathbf{C}$ in the z_{p+1} variable, with the z_i $(i \neq p+1)$ regarded as parameters which vary in some small polycylinder. As before we can assume that $\|A_{p+1}\|_{C^\epsilon}$ is as small as we like. Apply the implicit function theorem to get a solution F with $\|F\|_{C^{1+\epsilon}}$ small and which is analytic in $z_{p+1}, \overline{z}_{p+1}, \ldots, z_n, \overline{z}_n$ and holomorphic in the variables $z_1, \ldots, z_p$. This completes the induction, and the theorem for the real analytic case follows.

For the smooth case, we cannot apply the argument used in Step 2 to get an A_{p+1} holomorphic in $z_1, \ldots, z_p$. We will replace this with another induction argument. Define a proposition P_k for $1 \leq k \leq p$ as follows.

Proposition $\mathbf{P_k}$ *Let $\mathcal{G}^\infty \subset (\mathcal{C}^\infty)^d$ be a subsheaf over U_δ for some $\delta > 0$, satisfying condition (i)' of the theorem. Suppose that $\mathcal{G}^\infty$ is generated over U_δ by sections $r_1, \ldots, r_l$, which are holomorphic in $z_1, \ldots, z_p$. Let $a_1, \ldots, a_l$ be smooth functions on U_δ satisfying*

$$(\, \partial_{\overline{j}} a_1 \cdots \partial_{\overline{j}} a_l \,) \begin{pmatrix} r_1 \\ \vdots \\ r_l \end{pmatrix} = 0, \qquad \textit{for } j = 1, \ldots, k.$$

Then there exists δ' with $0 < \delta' < \delta$ and $(\, s_1 \cdots s_l \,)$ in the $\mathcal{C}^\infty$ sheaf of relations $\mathcal{R}(r_1, \ldots, r_l)$ over $U_{\delta'}$ with

$$\partial_{\overline{j}}(a_1 - s_1) = \ldots = \partial_{\overline{j}}(a_l - s_l) = 0, \qquad \textit{for } j = 1, \ldots, k.$$

We will show that P_k holds for $k = p$. Then since

$$\partial_{\overline{j}} A_{p+1} \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix} = 0, \qquad \text{for } j = 1, \ldots, p,$$

we can apply this result to each row of A_{p+1}. Hence, after making permissible changes to A_{p+1} and passing to a smaller open set, we can assume that A_{p+1} is holomorphic in $z_1, \ldots, z_p$. We will need to use the main inductive hypothesis in the proof of this induction.

Proof of $\mathbf{P_1}$ By assumption, the sheaf of relations $\mathcal{R}(r_1, \ldots, r_l)$ is locally finitely generated. Suppose the generators are

$$h_1 = (\, h_{11}\, h_{12} \cdots h_{1l}\,)\,, \ldots, h_m = (\, h_{m1}\, h_{m2} \cdots h_{ml}\,)\,.$$

Since $r_1, \ldots, r_l$ are holomorphic in z_1, and $\mathcal{R}(r_1, \ldots, r_l)$ satisfies *(i)'*, we can apply the *main* inductive hypothesis in the case $p = 1$ to this subsheaf. Hence we can assume that, after choosing δ' small enough, the generators $h_1, \ldots, h_m$ are holomorphic in the variable z_1. Now there exist smooth functions $t_1, \ldots, t_m$ on $U_{\delta'}$ satisfying

$$(\, \partial_{\overline{1}} a_1 \cdots \partial_{\overline{1}} a_l\,) = t_1 h_1 + \ldots + t_m h_m.$$

Let $\hat{t}_1, \ldots, \hat{t}_m$ be smooth functions satisfying $\partial_{\overline{1}} \hat{t}_i = t_i$ (shrinking δ' slightly.) Define

$$(\, s_1 \cdots s_l\,) = \hat{t}_1 h_1 + \ldots + \hat{t}_m h_m.$$

Then we see that $(\, s_1 \cdots s_l\,) \in \mathcal{R}(r_1, \ldots, r_l)$ and

$$\partial_{\overline{1}}(a_1 - s_1) = \ldots = \partial_{\overline{1}}(a_l - s_l) = 0,$$

completing the proof of P_1.

Proof of $\mathbf{P_k \Rightarrow P_{k+1}}$ We assume $1 \leq k \leq p-1$. Suppose that

$$(\, \partial_{\overline{j}} a_1 \cdots \partial_{\overline{j}} a_l\,) \begin{pmatrix} r_1 \\ \vdots \\ r_l \end{pmatrix} = 0, \qquad \text{for } j = 1, \ldots, k+1.$$

By the inductive hypothesis, we can suppose, after making some allowed changes and taking δ' small enough, that $\partial_{\overline{j}} a_i = 0$ for $j = 1, \ldots, k$ and $i = 1, \ldots, l$. As before let the generators of $\mathcal{R}(r_1, \ldots, r_l)$ be

$$h_1 = (\, h_{11}\, h_{12} \cdots h_{1l}\,)\,, \ldots, h_m = (\, h_{m1}\, h_{m2} \cdots h_{ml}\,)\,.$$

Since $r_1, \ldots, r_l$ are holomorphic in $z_1, \ldots, z_k$, by applying the main inductive hypothesis, we can assume that $h_1, \ldots, h_m$ are holomorphic in $z_1, \ldots, z_k$. There exist $t_1, \ldots, t_m$ smooth functions with

$$(\, \partial_{\overline{k+1}} a_1 \cdots \partial_{\overline{k+1}} a_l\,) = t_1 h_1 + \ldots + t_m h_m.$$

Notice that by applying $\partial_{\bar{j}}$ we get that

$$(\partial_{\bar{j}} t_1 \cdots \partial_{\bar{j}} t_m) \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} = 0, \qquad \text{for } j = 1, \ldots, k.$$

We can apply the *current* inductive hypothesis to see that, after shrinking δ', there exists $(s'_1 \cdots s'_m)$ in $\mathcal{R}(h_1, \ldots, h_m)$ such that

$$\partial_{\bar{j}}(t_1 - s'_1) = \ldots = \partial_{\bar{j}}(t_m - s'_m) = 0, \qquad \text{for } j = 1, \ldots, k.$$

By replacing the t_i by $t_i - s'_i$, we can then assume that $\partial_{\bar{j}} t_i = 0$ for $j = 1, \ldots, k$. Let $\hat{t}_i$ solve $\partial_{\overline{k+1}} \hat{t}_i = t_i$ for $i = 1, \ldots, m$ where the $\hat{t}_i$ are smooth and holomorphic in $z_1, \ldots, z_k$. Then, as before, set

$$(s_1 \cdots s_l) = \hat{t}_1 h_1 + \ldots + \hat{t}_m h_m.$$

Then $(s_1 \cdots s_l) \in \mathcal{R}(r_1, \ldots, r_l)$ and

$$\partial_{\bar{j}}(a_1 - s_1) = \ldots = \partial_{\bar{j}}(a_l - s_l) = 0, \qquad \text{for } j = 1, \ldots, k + 1.$$

This proves P_{k+1}.

It is now straightforward to complete the proof in the smooth case. To see that $\mathcal{F} = \ker \overline{\partial}|_{\mathcal{F}^\infty}$ is coherent and to establish

$$\mathcal{F} \otimes \mathcal{C}^\infty = \mathcal{F}^\infty,$$

one can use the result of Malgrange [Ma] on the flatness of $\mathcal{C}^\infty$ over $\mathcal{O}$, or apply Corollary 6.3.6 of [Ho].

3. The multiplier ideal sheaf

In this section, we give a proof of Theorem 2. Let H_k be a sequence of metrics as given in the hypothesis of the theorem, defining the multiplier ideal sheaf $\mathcal{F}_R$ over $\tilde{X}$. We now prove the differential inclusion relation.

Proposition 3.1 *$\overline{\partial} \mathcal{F}_R \subseteq \mathcal{F}_R \otimes T^{0,1}$.*

Proof Let x be in $\tilde{X}$, and let U be an open set in $\tilde{X}$ containing x. Choose V compact and W open such that $x \in W \subset V \subset U$ and let ψ be a smooth cut-off function supported in V and equal to 1 in W. Suppose that s satisfies

$$\int_U |s|^2_{H_k} \to 0 \quad \text{as } k \to \infty.$$

Then

$$\begin{aligned}
\int_W |\overline{\partial} s|^2_{H_k} &= -i \int_W \Lambda \langle \overline{\partial} s, \overline{\partial} s \rangle_{H_k} \\
&\leq -i \int_X \Lambda \langle \psi h_k \overline{\partial} s, \overline{\partial} s \rangle_{H_0} \\
&= -i \int_X \Lambda \langle \psi (\partial_0 h_k) \overline{\partial} s, s \rangle_{H_0} - i \int_X \Lambda \langle \psi h_k \partial_0 \overline{\partial} s, s \rangle_{H_0} \\
&\quad - i \int_X \Lambda \langle (\partial \psi) h_k \overline{\partial} s, s \rangle_{H_0} \\
&= -i \int_X \Lambda \langle \psi (h_k)^{-\frac{1}{2}} (\partial_0 h_k) \overline{\partial} s, (h_k)^{\frac{1}{2}} s \rangle_{H_0} \\
&\quad - i \int_X \Lambda \langle \psi (h_k)^{\frac{1}{2}} \partial_0 \overline{\partial} s, (h_k)^{\frac{1}{2}} s \rangle_{H_0} \\
&\quad - i \int_X \Lambda \langle (\partial \psi) (h_k)^{\frac{1}{2}} \overline{\partial} s, (h_k)^{\frac{1}{2}} s \rangle_{H_0} \\
&\leq C \Big(\int_V |(h_k)^{-\frac{1}{2}} \partial_0 h_k|^2_{H_0} \Big)^{\frac{1}{2}} \Big(\int_U |s|^2_{H_k} \Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_V |\partial_0 \overline{\partial} s|^2_{H_k} \Big)^{\frac{1}{2}} \Big(\int_U |s|^2_{H_k} \Big)^{\frac{1}{2}} \\
&\quad + C \Big(\int_V |\overline{\partial} s|^2_{H_k} \Big)^{\frac{1}{2}} \Big(\int_U |s|^2_{H_k} \Big)^{\frac{1}{2}}
\end{aligned}$$

where C denotes a constant that does not depend on k. The last two terms in the last line clearly tend to zero as k tends to zero. To see that the first term in the last line also tends to zero, observe that

$$\begin{aligned}
\int_X |(h_k)^{-\frac{1}{2}} \partial_0 h_k|^2_{H_0} &= i \int_X \Lambda \langle (h_k)^{-1} \partial_0 h_k, \partial_0 h_k \rangle_{H_0} \\
&= i \int_X \Lambda \langle \overline{\partial} ((h_k)^{-1} \partial_0 h_k), h_k \rangle_{H_0} \\
&= \int_X \langle (i \hat{F}_{H_k} - i \hat{F}_{H_0}), h_k \rangle_{H_0} \\
&\leq C
\end{aligned}$$

where we have used the fact that the contracted curvature $i\hat{F}_{H_k}$ is bounded from above. This completes the proof.

In order to apply Theorem 1 we must see that $\mathcal{F}_R$ is locally finitely generated. This follows from the next lemma, which gives an alternative description of the multiplier ideal sheaf.

Lemma 3.2 *Define a sheaf $\mathcal{S}_\mathcal{R}$ on $\tilde{X}$ by*

$$\mathcal{S}_\mathcal{R}(U) = \{s \in \mathcal{E}_R(U) \mid h_\infty s = 0\}.$$

Then $\mathcal{F}_R = \mathcal{S}_\mathcal{R}$.

Proof Let x be in $\tilde{X}$, and let $x \in U \subset \tilde{X}$. Choose V compact and W open such that $x \in W \subset V \subset U$. If s is in $\mathcal{S}_\mathcal{R}(U)$ then

$$\begin{aligned} \int_W |s|^2_{H_k} &= \int_W \langle (h_k - h_\infty)s, s\rangle_{H_0} \\ &\leq \|h_k - h_\infty\|_{C^0(V)} \int_V |s|^2_{H_0} \\ &\to 0, \end{aligned}$$

as k tends to infinity. This proves $\mathcal{S}_\mathcal{R} \subset \mathcal{F}_R$. On the other hand, suppose that

$$\int_U |s|^2_{H_k} \to 0, \qquad \text{as } k \to \infty.$$

Then for any smooth section t of E over U,

$$\begin{aligned} \int_V \langle h_\infty s, t\rangle_{H_0} &= \int_V \langle (h_\infty - h'_k)s, t\rangle_{H_0} + \int_V \langle s, t\rangle_{H_k} \\ &\leq \|h_\infty - h'_k\|_{C^0(V)} (\int_V |s|^2_{H_0})^{\frac{1}{2}} (\int_V |t|^2_{H_0})^{\frac{1}{2}} + (\int_U |s|^2_{H_k})^{\frac{1}{2}} (\int_V |t|^2_{H_k})^{\frac{1}{2}} \\ &\to 0 \end{aligned}$$

as k tends to infinity. Hence $h_\infty s = 0$ on W and the lemma is proved.

Since h_∞ is real analytic, $\mathcal{F}_R$ is locally the kernel of a homomorphism between coherent sheaves of $\mathcal{O}_R$ modules, and so $\mathcal{F}_R$ is locally finitely generated. Applying Theorem 1 completes the proof.

4. Hermitian Einstein metrics on stable bundles

For this section, we will assume the hypotheses of Theorem 3. The background material for this section can be found in [D2] and [DoKr] (see also [S2] for a good exposition.)

Let H_0 be a *real analytic* Hermitian metric on the fibres of E. We can obtain such a metric in the following way. Tensor E with the hyperplane line bundle L raised to a high power K so that the holomorphic sections of $E \otimes L^K$ embed X into a Grassmannian $G(r, M)$ of r-planes in $\mathbf{C}^M$, where M is the dimension of $H^0(X, E \otimes L^K)$. Then, via this embedding, pull back the canonical metric on the universal bundle $U(r)$ over the Grassmannian to get a real analytic metric on $E \otimes L^K$. Then use the canonical real analytic metric on L to get a real analytic metric H_0 on E.

Let λ be the average of $(\mathrm{Tr}\, \hat{F}_0)/r$. We consider the flow of endomorphisms h_t with $h_0 = Id_E$ given by the flow

$$\frac{\partial h_t}{\partial t} = -2ih_t(\hat{F}_t - \lambda I)$$

introduced by Donaldson [D2] (F_t is the curvature of the metric $H_t = H_0 h_t$.) We call this the Donaldson heat flow. It is shown in [D2] that solutions exist for all time. The flow is, roughly speaking, the same as the Yang-Mills flow up to gauge. In particular, if A_0 is the unitary connection associated to H_0 and the holomorphic structure on E, and if g_t in the complexified gauge group $\mathcal{G}^C$ satisfies $g_t^* g_t = h_t$ then $A_t = g_t(A_0)$ satisfies the gauge equivalent Yang-Mills flow

$$\frac{\partial A_t}{\partial t} = -d_{A_t}^* F_{A_t} + d_{A_t}(\alpha(t))$$

for $\alpha(t) \in \Omega^0(\mathbf{g}_E)$.

Make a (real analytic) conformal change to H_0 so that $\det h_t = 1$ along the flow. Also, normalize the metric on X so that $\mathrm{Vol}\,(X) = 2\pi$.

Suppose first that $\sup_X |h_t|_{H_0}$ is bounded uniformly in t. By the argument in [D2], we have a C^1 bound and L_2^p bounds for all p for h_t. Hence, for a sequence of times $t_k \to \infty$, $H_k = H_{t_k}$ converges strongly in L_1^p to a Hermitian metric H_∞, which is itself in L_2^p. Then

$$\hat{F}_{H_k} - \lambda I \rightharpoonup \hat{F}_{H_\infty} - \lambda I$$

weakly in L^p. Now, Donaldson's functional (see [D2]) is bounded below when E is stable (in fact, semi-stable) and calculating the functional along the flow gives

$$\hat{F}_{H_k} - \lambda I \to 0$$

strongly in L^2. Hence we get that $\hat{F}_\infty = \lambda I$ and by elliptic estimates, H_∞ is a smooth (and in fact real analytic) Hermitian-Einstein metric.

Suppose then that the C^0 norm of h_t is unbounded along the flow. We will show that this leads to a contradiction. Take a sequence h_{t_k} with the C^0 norm tending to infinity as k tends to infinity. Then define normalized endomorphisms h'_k by

$$h'_k = \frac{h_{t_k}}{\sup_X |h_{t_k}|_{H_0}},$$

and let $H'_k = H_0 h'_k$. Then we have the following theorem, the proof of which is, for the most part, contained in [D2] and [DoKr].

Theorem 4.1 *There exists a finite set of points $\{x_1, \ldots, x_p\} \subset X$ such that, after passing to a subsequence, h'_k converges uniformly on compact sets in $X \backslash \{x_1, \ldots, x_p\}$ to a real analytic endomorphism h_∞.*

Proof Define $A_k = h_k'^{1/2}(A_0)$. Then we have

(i) F_{A_k} are bounded in L^2;

(ii) $\hat{F}_{A_k}$ are uniformly bounded;

(iii) $\nabla_{A_k} \hat{F}_{A_k} \to 0$ in L^2.

(i) follows from the fact that the Yang-Mills flow is the gradient flow for the Yang-Mills functional, (ii) follows from a maximum principle argument (see [D2]) and a proof of (iii) is given in [DoKr] (Proposition 6.2.14).

It is shown in [DoKr] that there exists a set of points $\{x_1, \ldots, x_p\}$ in X and a cover of $X \backslash \{x_1, \ldots, x_p\}$ by a system of balls such that, after passing to a subsequence, each connection A_k has curvature with L^2 norm less than a small $\epsilon > 0$. If this ϵ is small enough, then by a theorem of Uhlenbeck's [Uh], we can put the connections in Coulomb gauge over these balls. By elliptic estimates and a standard patching argument (see [DoKr] for details), there exist gauge transformations v_k over the punctured manifold such that $v_k h_k'^{1/2}(A_0)$ are bounded in L^2_2. Put $g_k = v_k h_k'^{1/2}$, and let $\tilde{A}_k = g_k(A_0)$. Then if $\tilde{A}_k^{1,0}$ and $\tilde{A}_k^{0,1}$ are the $(1,0)$ and $(0,1)$ components of $\tilde{A}_k$ we have

$$\begin{aligned} \tilde{A}_k^{1,0} &= A_0^{1,0} + g_k^{-1}(\partial_{A_0} g_k) \\ \tilde{A}_k^{0,1} &= A_0^{0,1} + (\overline{\partial} g_k) g_k^{-1}. \end{aligned}$$

Since $g_k^* g_k = h'_k$ we have that the g_k are bounded uniformly, and from the above, they are bounded in L^2_3. Hence, after passing to a subsequence, the g_k

converge uniformly on compact subsets to g_∞, from which it follows that h'_k converges uniformly on compact subsets to h_∞. It remains to see that h_∞ is real analytic. Now $\tilde{A}_k$ converges over the punctured manifold to A_∞, weakly in L^2_2, and by condition (iii) we must have

$$\nabla_{A_\infty} \hat{F}_{A_\infty} = 0,$$

which is elliptic in a Coulomb gauge. The elliptic estimates give that A_∞ is smooth. Since the coefficients of the equation are real analytic (using the fact that ω is real analytic), it follows that A_∞ is real analytic. Now

$$\begin{aligned} A^{1,0}_\infty &= A^{1,0}_0 + g^{-1}_\infty(\partial_{A_0} g_\infty) \\ A^{0,1}_\infty &= A^{0,1}_0 + (\overline{\partial} g_\infty) g^{-1}_\infty \end{aligned}$$

and A_0 is itself real analytic. It follows that g_∞ is real analytic, and therefore h_∞ is also.

Now let $\mathcal{F}_R$ be the multiplier ideal sheaf associated to the sequence of metrics $\{H'_k\}$ over $\tilde{X} = X\backslash\{x_1, \ldots, x_p\}$. Then, since $\hat{F}$ is bounded along the flow, and by the above, we can apply Theorem 2 to get a coherent sheaf $\mathcal{F} \subset \mathcal{E}$ over $\tilde{X}$. In fact, this defines a coherent subsheaf of E over the whole of X. To see this, recall that a coherent subsheaf may be defined in the following way (see [UY], section 7, for example.) Let U_i be a cover of X so that $E|_{U_i}$ is trivial and let t_{ij} be the transition functions of E defined on the intersections $U_i \cap U_j$. Then a coherent subsheaf of E of rank k can be defined to be a set $\{f_i\}$ of *rational maps* from U_i to the Grassmannian $G(k, r)$ with transformation maps t_{ij} on the overlaps. Thus, in particular, we can extend over a finite number of points in X.

We need the following lemma.

Lemma 4.2 *There exists a positive constant c, independent of k, such that*

$$\|\mathrm{Tr}\, h_k\|_{L^1(X)} \geq c \sup_X \mathrm{Tr}\, h_k.$$

Proof We use the inequality

$$\triangle \log \mathrm{Tr}\, h_k \geq -\frac{1}{2}(|\hat{F}_{H_0}| + |\hat{F}_{H_k}|)$$

from [S2], and the fact that $\hat{F}$ is uniformly bounded along the flow. We know that the Green's function $G = G(x, y)$ is bounded below by some constant $-A$. Then

$$\begin{aligned}
\log \operatorname{Tr} h_k(x) &= \frac{1}{2\pi} \int_X \log \operatorname{Tr} h_k(y) dV(y) \\
&\quad + \int_X (-\triangle \log \operatorname{Tr} h_k)(y)(G(x, y) + A) dV(y) \\
&\leq \frac{1}{2\pi} \int_X \log \operatorname{Tr} h_k(y) dV(y) + C \\
&\leq \log(\frac{1}{2\pi} \int_X \operatorname{Tr} h_k(y) dV(y)) + C,
\end{aligned}$$

where C is independent of k. This proves the lemma.

Since $\det h_k = 1$ along the flow, the above shows that at least one eigenvalue of h_k must tend to infinity on a non-empty open set of X. Hence

$$0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E}.$$

We will now show that $\mathcal{F}$ is destabilizing, which will give the contradiction. Note that $\mathcal{F}$ is a subbundle outside a subvariety Z of codimension 1. We will calculate the second fundamental form, following the method of [UY]. First observe that given a semi-positive self-adjoint (with respect to H_0) endomorphism h, one can define an endomorphism h^σ by diagonalizing h at a point in a unitary frame with respect to H_0 and then raising each matrix entry to the power σ. Then note that outside $Z \subset X$, the pointwise limit

$$\pi = \lim_{\sigma \to 0} (I - h_\infty^\sigma)$$

is the orthogonal projection of E onto the subbundle defined by $\mathcal{F}$. Define the *slopes* $\mu(\mathcal{F})$ and $\mu(\mathcal{E})$ by

$$\mu(\mathcal{E}) = \frac{i}{2\pi} \frac{\int_X \operatorname{Tr}(\hat{F}_0)}{\operatorname{rank}(\mathcal{E})} \qquad \text{and} \qquad \mu(\mathcal{F}) = \frac{i}{2\pi} \frac{\int_X \operatorname{Tr}(\hat{F}_0(\mathcal{F}))}{\operatorname{rank}(\mathcal{F})}.$$

Note that $\mu(\mathcal{E}) = i\lambda$. The following lemma completes the proof of Theorem 3.

Lemma 4.3

$$\mu(\mathcal{F}) \geq \mu(\mathcal{E}).$$

Proof First observe that for $0 < \sigma \le 1$,

$$\begin{aligned}
\int_X |dh_k'^{\sigma}|^2_{H_0} &= 2\int_X |\partial_0 h_k'^{\sigma}|^2_{H_0} \\
&\le \int_X |(h_k')^{-\sigma/2}(\partial_0 h_k'^{\sigma})|^2_{H_0} \\
&\le \int_X \langle h_k^{-1}(\partial_0 h_k), \partial_0 h_k'^{\sigma}\rangle_{H_0} \\
&= \int_X i\langle (\hat{F}_k - \hat{F}_0), h_k'^{\sigma}\rangle_{H_0} \\
&\le C
\end{aligned}$$

where in the third line, we have used the pointwise inequality

$$|h^{-\sigma/2}(\partial_0 h^{\sigma})|^2_{H_0} \le \langle h^{-1}(\partial_0 h), \partial_0 h^{\sigma}\rangle_{H_0} \tag{4.1}$$

for $0 < \sigma \le 1$ and h a positive self-adjoint endomorphism (see [UY], Lemma 4.1). Hence

$$I - h_k'^{\sigma} \rightharpoonup \tilde{\pi}$$

weakly in L^2_1, for some $\tilde{\pi}$ in L^2_1, taking the limit as k tends to infinity and then as σ tends to zero. Notice that π and $\tilde{\pi}$ must agree on $X\backslash Z$. Then

$$\begin{aligned}
i\int_X \mathrm{Tr}\,\hat{F}_0(\mathcal{F}) &= i\int_X \mathrm{Tr}\,(\hat{F}_0\pi) - \int_X |\partial_0\pi|^2_{H_0} \\
&= i\int_X \mathrm{Tr}\,((\hat{F}_0 - \lambda I)\pi) + \int_X \mathrm{Tr}\,(\mu(\mathcal{E})\pi) - \int_X |\partial_0\pi|^2_{H_0} \\
&= i\int_X \mathrm{Tr}\,((\hat{F}_0 - \lambda I)\pi) - \int_X |\partial_0\pi|^2_{H_0} + \mathrm{Vol}\,(X)\,\mathrm{rank}\,(\mathcal{F})\mu(\mathcal{E}).
\end{aligned}$$

Hence $\mu(\mathcal{F}) \ge \mu(\mathcal{E})$ if and only if

$$i\int_X \mathrm{Tr}\,((\hat{F}_0 - \lambda I)\pi) \ge \int_X |\partial_0\pi|^2_{H_0}.$$

First, observe that using the inequality (4.1),

$$\begin{aligned}
i\int_X \mathrm{Tr}\,((\hat{F}_0 - \lambda I)h_k'^{\sigma}) &= i\int_X \mathrm{Tr}\,((\hat{F}_k - \lambda I)h_k'^{\sigma}) - \int_X \langle \overline{\partial}(h_k^{-1}(\partial_0 h_k)), h_k'^{\sigma}\rangle_{H_0} \\
&\le i\int_X \mathrm{Tr}\,((\hat{F}_k - \lambda I)h_k'^{\sigma}) - \int_X |\partial_0 h_k'^{\sigma}|^2_{H_0}.
\end{aligned}$$

Then by lower semicontinuity,

$$\begin{aligned}
\int_X |\partial_0 \pi|^2_{H_0} &\leq \liminf_{\sigma\to 0} \liminf_{k\to\infty} \int_X |\partial_0 (I - h'^{\sigma}_k)|^2_{H_0} \\
&= \liminf_{\sigma\to 0} \liminf_{k\to\infty} \int_X |\partial_0 h'^{\sigma}_k|^2_{H_0} \\
&\leq \liminf_{\sigma\to 0} \liminf_{k\to\infty} (-i \int_X \mathrm{Tr}\,((\hat{F}_0 - \lambda I) h'^{\sigma}_k) + i \int_X \mathrm{Tr}\,((\hat{F}_k - \lambda I) h'^{\sigma}_k)) \\
&= i \int_X \mathrm{Tr}\,((\hat{F}_0 - \lambda I)\pi) + \liminf_{\sigma\to 0} \liminf_{k\to\infty} i \int_X \mathrm{Tr}\,((\hat{F}_k - \lambda I) h'^{\sigma}_k),
\end{aligned}$$

using the fact that $\int_X \mathrm{Tr}\,(\hat{F}_0 - \lambda I) = 0$ and $I - h'^{\sigma}_k \rightharpoonup \tilde{\pi}$ weakly in L^2_1. Now since the Donaldson functional is bounded below, we know that $\int_X |\hat{F}_k - \lambda I|^2_{H_k}$ tends to zero as k tends to infinity. This gives the desired inequality and completes the proof.

Acknowledgements: The author would like to thank his thesis advisor, D.H. Phong for suggesting the problem, and for his constant advice and encouragement. The author is also indebted to Y-T. Siu, who visited Columbia University in the Fall of 2002 and gave a series of lectures [S5] on multiplier ideal sheaves. The ideas presented in those talks and in private discussions were invaluable in the shaping of this paper. In particular, Professor Siu proposed the use of a complex Frobenius theorem. The author is also grateful to Jacob Sturm for some very helpful suggestions, and to Zhiqin Lu for some useful discussions. This paper will form part of the author's forthcoming PhD thesis at Columbia University.

References

[Bo] Bombieri, E. *Algebraic values of meromorphic maps*, Invent. Math. 10 (1970), 267-287 and Addendum, Invent. Math., 11 (1970), 163-166

[DT] de Bartolomeis, P. and Tian, G. *Stability of complex vector bundles*, J. Diff. Geom. (2), 43 (1996), 231-275

[DC] de Cataldo, M.A.A. *Singular hermitian metrics on vector bundles*, J. reine angew. Math., 502 (1998), 93-122

[De] Demailly, J-P. *Effective bounds for very ample line bundles*, Invent. Math., 124 (1996), 243-261

[DeKo] Demailly, J-P. and Kollár, J. *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4), 34 (2001), 525-556

[D1] Donaldson, S.K. *A new proof of a theorem of Narasimhan and Seshadri*, J. Diff. Geom., 18 (1983), 269-277

[D2] Donaldson, S.K. *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. (3), 50 (1985), 1-26

[DoKr] Donaldson, S.K. and Kronheimer, P.B. *The geometry of four manifolds*, Clarendon Press, Oxford (1990)

[Ho] Hörmander, L. *An introduction to complex analysis in several variables*, North-Holland, Amsterdam, 3rd Edition (1990)

[Ko] Kohn, J.J. *Subellipticity of the* $\overline{\partial}$*-Neumann problem on pseudo-convex domains: sufficient conditions*, Acta Math., 142 (1979), 79-122

[KM] Koszul, J-L. and Malgrange, B. *Sur certaines structures fibrés complexes*, Arch. Math., 9 (1958), 102-109

[Ma] Malgrange, B. *Ideals of differentiable functions*, Oxford University press (1966)

[Na] Nadel, A.M. *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math., 132 (1990), 549-596

[NS] Narasimhan, M.S. and Seshadri, C.S. *Stable and unitary vector bundles on compact Riemann surfaces*, Ann. of Math., 82 (1965), 540-567

[NN] Newlander, A. and Nirenberg, L. *Complex analytic co-ordinates in almost complex manifolds*, Ann. of Math., 65 (1957), 391-404

[Pa] Pali, N. *Faisceaux* $\overline{\partial}$*-cohérents sure les variétés complexes*, preprint, arXiv: math.AG/0301146 (2003)

[Si] Simpson, C.T. *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. (4), 1 (1988), 867-918

[S1] Siu, Y-T. *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math., 27 (1974), 53-156

[S2] Siu, Y-T. *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, Birkhäuser Verlag, Basel (1987)

[S3] Siu, Y-T. *An effective Matsusaka big theorem*, Ann. Inst. Fourier, 43 (1993), 1387-1405

[S4] Siu, Y-T. *Effective very ampleness*, Invent. Math. 124 (1996), 563-571

[S5] Siu, Y-T. *Multiplier ideal sheaves in algebraic and complex geometry*, Samuel Eilenberg Lectures at Columbia University (2002), unpublished

[Sk] Skoda, H. *Sous-ensembles analytiques d'ordre fini ou infini dans* $\mathbf{C}^n$, Bull. Soc. Math. France, 100 (1972), 353-408

[Uh] Uhlenbeck, K.K. *Connections with* L^p *bounds on curvature*, Comm. Math. Phys., 83 (1982), 31-42

[UY] Uhlenbeck, K.K. and Yau, S.T. *The existence of Hermitian Yang-Mills connections on stable bundles over Kähler manifolds*, Comm. Pure & Applied Math., 36 (1986), 257-293